

DP Analysis & Approaches SL / HL

Maths with Lemon

Lesson 9.2 Integration by Substitution

Mathematician: _____

SL 5.10	Guidance, clarification and syllabus links
The composites of any of these with the linear function $ax + b$.	Example: $f'(x) = \cos(2x + 3) \Rightarrow f(x) = \frac{1}{2} \sin(2x + 3) + C$
Integration by inspection (reverse chain rule) or by substitution for expressions of the form: $\int k g'(x) f(g(x)) dx$	Examples: $\int 2x(x^2 + 1)^4 dx, \quad \int 4x \sin(x^2) dx, \quad \int \frac{\sin x}{\cos x} dx$

Warm Up

Find the indefinite integral of each of the following.

a) $f(x) = 5x^7$	b) $g(x) = 8x^{3/2} - 2 + \frac{1}{x^2}$	c) $h(x) = \frac{1}{x^3} + \frac{1}{x}$
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Integration by Substitution (Reverse Chain Rule)

The chain rule gives us that

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

We can also work backwards if we correctly identify the functions that are present:

$$\int f'(g(x)) \cdot g'(x) dx = f(g(x)) + C.$$

Example 1

$$\int (5x + 4)^5 dx$$

Step 1: Set the inner function as u .

Step 2: Find the derivative of u .

Step 3: Solve for dx .

Step 4: Substitute in terms of u and du .

Step 5: Integrate with respect to u .

Step 6: Substitute back to terms of x .

Example 2

$$\int (3x + 2)^4 dx$$

Example 3

$$\int 3t^2(t^3 + 4)^5 dt$$

Example 4

$$\int \sqrt{4x - 5} dx$$

Example 5

$$\int t^2(t^3 + 4)^{-1/2} dt$$

Example 6

$$\int \cos(2x + 1) dx$$

Example 7

$$\int \sin^{10} x \cos x \, dx$$

Example 8

$$\int \frac{(\sqrt{x} - 1)^2}{\sqrt{x}} \, dx$$

Example 9

$$\int \sqrt{x^3 + x^2} (3x^2 + 2x) \, dx$$

Example 10

$$\int e^{-x/\pi + 1/3} \, dx$$

Example 11

Given that

$$\frac{dy}{dx} = x \sin(3x^2),$$

find an expression for y in terms of x .

Example 12

Find

$$\int \left(\frac{e^{2x} - e^{-2x}}{e^x} \right) dx.$$

Example 13

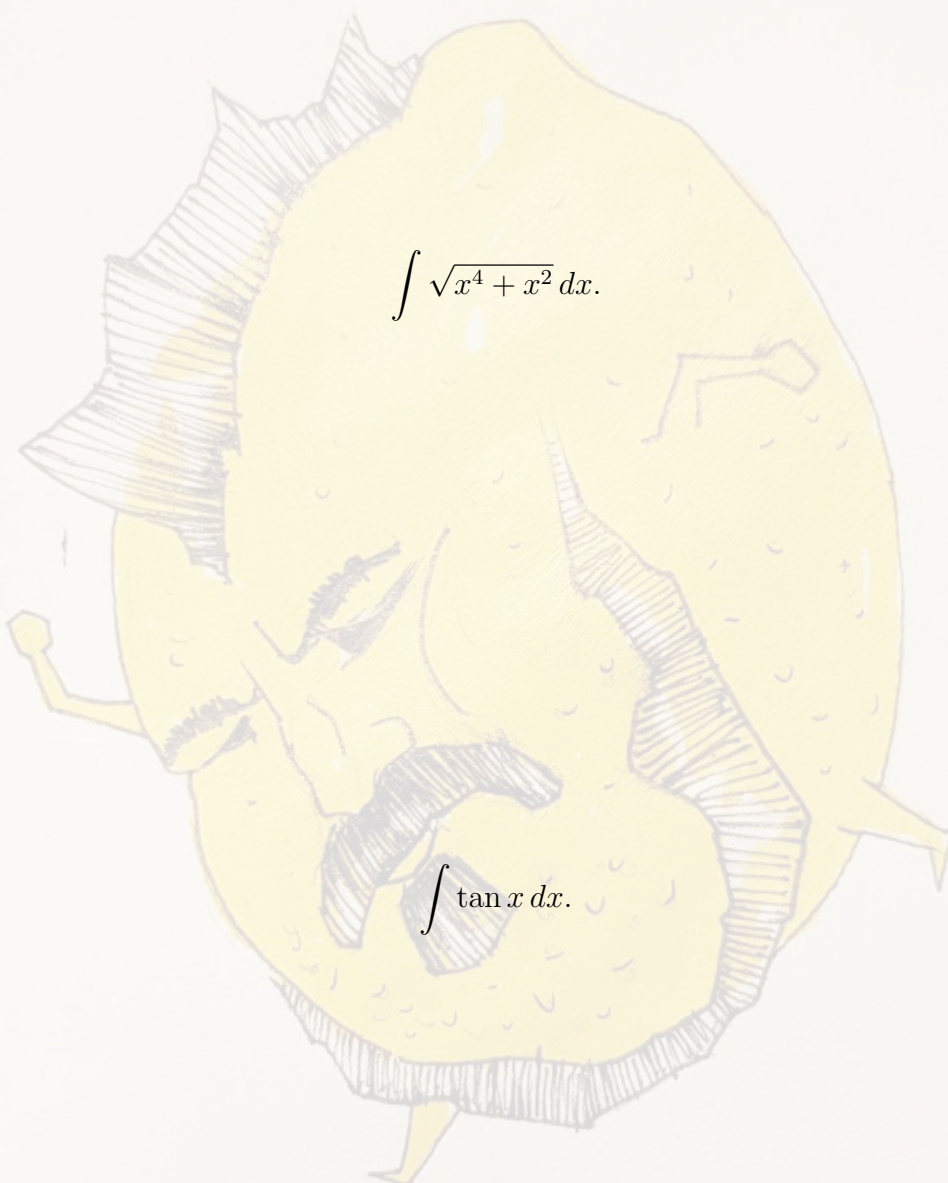
Find

$$\int \sqrt{x^4 + x^2} dx.$$

Example 14

Find

$$\int \tan x dx.$$



Solutions

Lesson 9.2 Integration by Substitution

Warm Up

a)

$$\int 5x^7 dx = 5 \left(\frac{x^8}{8} \right) + C = \boxed{\frac{5}{8}x^8 + C}.$$

b)

$$\begin{aligned} \int (8x^{3/2} - 2 + x^{-2}) dx \\ = 8 \left(\frac{x^{5/2}}{5/2} \right) - 2x - \frac{1}{x} + C \\ = \boxed{\frac{16}{5}x^{5/2} - 2x - \frac{1}{x} + C}. \end{aligned}$$

c)

$$\begin{aligned} \int (x^{-3} + x^{-1}) dx = -\frac{1}{2x^2} + \ln|x| + C. \\ \boxed{-\frac{1}{2x^2} + \ln|x| + C} \end{aligned}$$

Examples 1–3

Example 1

$$u = 5x + 4, \quad du = 5 dx, \quad dx = \frac{1}{5} du.$$

Therefore

$$\begin{aligned} \int (5x + 4)^5 dx = \frac{1}{5} \int u^5 du = \frac{u^6}{30} + C. \\ \boxed{\frac{(5x + 4)^6}{30} + C} \end{aligned}$$

Example 2

$$u = 3x + 2, \quad du = 3 dx.$$

$$\int (3x + 2)^4 dx = \frac{1}{3} \int u^4 du = \boxed{\frac{(3x + 2)^5}{15} + C}.$$

Example 3

$$u = t^3 + 4, \quad du = 3t^2 dt.$$

Hence

$$\int 3t^2(t^3 + 4)^5 dt = \int u^5 du = \boxed{\frac{(t^3 + 4)^6}{6} + C}.$$

Examples 4–8

Example 4

$$u = 4x - 5, \quad du = 4dx.$$

$$\int \sqrt{4x - 5} \, dx = \frac{1}{4} \int u^{1/2} du = \boxed{\frac{1}{6}(4x - 5)^{3/2} + C}.$$

Example 5

$$u = t^3 + 4, \quad du = 3t^2 dt.$$

$$\int t^2(t^3 + 4)^{-1/2} dt = \frac{1}{3} \int u^{-1/2} du = \boxed{\frac{2}{3}\sqrt{t^3 + 4} + C}.$$

Example 6

$$u = 2x + 1, \quad du = 2dx.$$

$$\int \cos(2x + 1) dx = \frac{1}{2} \int \cos u \, du = \boxed{\frac{1}{2} \sin(2x + 1) + C}.$$

Example 7

$$u = \sin x, \quad du = \cos x \, dx.$$

$$\int \sin^{10} x \cos x \, dx = \int u^{10} du = \boxed{\frac{\sin^{11} x}{11} + C}.$$

Example 8

Let

$$u = \sqrt{x} - 1.$$

Then

$$du = \frac{1}{2\sqrt{x}} dx \quad \Rightarrow \quad \frac{dx}{\sqrt{x}} = 2du.$$

Thus

$$\int \frac{(\sqrt{x} - 1)^2}{\sqrt{x}} dx = 2 \int u^2 du = \frac{2}{3} u^3 + C.$$

$$\boxed{\frac{2}{3}(\sqrt{x} - 1)^3 + C}$$

Examples 9–12

Example 9

Let

$$u = x^3 + x^2.$$

Then

$$du = (3x^2 + 2x)dx.$$

Therefore

$$\int \sqrt{x^3 + x^2}(3x^2 + 2x)dx = \int u^{1/2}du = \boxed{\frac{2}{3}(x^3 + x^2)^{3/2} + C}.$$

Example 10

Let

$$u = -\frac{x}{\pi} + \frac{1}{3}.$$

Then

$$du = -\frac{1}{\pi}dx \Rightarrow dx = -\pi du.$$

Hence

$$\int e^{-x/\pi+1/3}dx = -\pi \int e^u du = \boxed{-\pi e^{-x/\pi+1/3} + C}.$$

Example 11

Since

$$\frac{dy}{dx} = x \sin(3x^2),$$

we have

$$y = \int x \sin(3x^2)dx.$$

Let

$$u = 3x^2, \quad du = 6x dx.$$

Then

$$y = \frac{1}{6} \int \sin u du = -\frac{1}{6} \cos u + C.$$

Therefore

$$\boxed{y = -\frac{1}{6} \cos(3x^2) + C}.$$

Example 12

First simplify:

$$\frac{e^{2x} - e^{-2x}}{e^x} = e^x - e^{-3x}.$$

Therefore

$$\int (e^x - e^{-3x})dx = e^x + \frac{1}{3}e^{-3x} + C.$$

$$\boxed{e^x + \frac{1}{3}e^{-3x} + C}$$

Examples 13–14

Example 13

$$\sqrt{x^4 + x^2} = \sqrt{x^2(x^2 + 1)} = |x|\sqrt{x^2 + 1}.$$

For $x \geq 0$,

$$\int \sqrt{x^4 + x^2} dx = \int x\sqrt{x^2 + 1} dx.$$

Let

$$u = x^2 + 1, \quad du = 2x dx.$$

Then

$$\frac{1}{2} \int u^{1/2} du = \frac{1}{3} u^{3/2} + C.$$

Thus

$$\boxed{\frac{1}{3}(x^2 + 1)^{3/2} + C}, \quad x \geq 0.$$

For $x < 0$,

$$\boxed{-\frac{1}{3}(x^2 + 1)^{3/2} + C}.$$

Example 14

Rewrite

$$\tan x = \frac{\sin x}{\cos x}.$$

Let

$$u = \cos x, \quad du = -\sin x dx.$$

Then

$$\int \tan x dx = - \int \frac{1}{u} du = -\ln |u| + C.$$

Therefore

$$\boxed{-\ln |\cos x| + C}$$

or equivalently

$$\boxed{\ln |\sec x| + C}.$$



Maths with Lemon

Mathematics made clear.