DP Analysis & Approaches (SL/HL)

Maths with Lemon

Lesson 1.4HL Advanced Counting Principles

Mathematician: ________________________________

SL 1.9	Guidance, clarification and syllabus links
	Use of Pascal's triangle and ${}^nC_r$. ${}^nC_r$ should be found using both the formula and technology. Example: Find r when ${}^6C_r = 20$, using a table of values generated with technology.

Warm Up

1) A twenty-sided die is rolled. Calculate the number of results that are:
(a) divisible by 2 and divisible by 3
(b) divisible by 2 or divisible by 3.

2) Angel wants to set up a new username consisting of the five letters A, N, G, E, L, followed by the three digits 1, 2, 3. Find the number of ways she can do this.

Permutation and Combination Problem Solving Techniques

These strategies may help you solve more complex problems involving permutations and combinations:

- **Count what you do not want** and subtract that from the total to get what you do want.
  Example: Find how many numbers between 1 and 100 are not divisible by five.
- **Treat any items that must be together as a single item** in the list.
  Example: Find the number of permutations of ABCDEFG where A and B must be together.
  Note: Remember that the items that are together also need to be permuted.

Applying Combination Solving Techniques

Ina, Henry, Kevin, Angel, Richard, Edward, and Nemo line up for a netball team photo. Find the number of possible arrangements in which:
(a) Ina and Angel are next to each other.

(b) Ina and Angel are not next to each other.

Another Problem-Solving Technique

- To separate three or more items, consider the **gaps** they can fit into.

Example: How many ways can ABCDEFGH be ordered so that A, B, and C are not next to each other?

Separating Elements

Ina, Henry, Kevin, Angel, Richard, Edward, and Nemo line up for a netball team photo. Find the number of possible arrangements in which Richard, Henry, and Kevin are not near each other.

Example 9

Six students from AAHL, four students from AASL, and three students from AIHL all arrive to line up at lunch. Find:

(a) The number of possible ways they can arrange themselves given that the students from AIHL must be separated.

(b) The number of possible ways in which the students from AIHL are grouped together.

Higher Level Combinatorics Challenge Problems

Problem 1

Find the number of permutations of the word ‘CHANGED’ that start and end with a consonant.

Problem 2

A committee of five students is to be selected from a class of 28. The two youngest cannot both be on the committee. Find the number of ways the committee can be selected.

Problem 3

A group of 19 students contains nine boys and ten girls. A team is said to be mixed if it contains at least one boy and at least one girl. Find the number of ways a mixed team of seven can be chosen from the 19 students.

Problem 4

A class consists of six girls and nine boys. A committee of six is chosen.

(a) How many possible committees can be chosen if there are no constraints?

(b) How many different committees are possible if

(i) it must not contain John

(ii) it must contain at least two girls?

(c) If the committee is chosen at random, what is the probability that it contains at least two girls?

Problem 5

A set of positive integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is used to form a pack of nine cards. Each card displays one positive integer without repetition from this set. Grace wishes to select four cards at random.

(a) Find the number of selections Grace could make if the largest integer drawn among the four cards is either a 5, a 6 or a 7.

(b) Find the number of selections Grace could make if at least two of the four integers drawn are even.

Problem 6

In a doctor's waiting room, there are 10 seats in a row. Six people are waiting to be seen.

(a) Find the number of ways they can be seated.

(b) One of them has a bad cold and mustn't sit next to anyone else. Find the number of ways the six people can be seated now.

Problem 7

Twelve friends arrive at a quiz but are told that the maximum number in a team is six. Find the number of ways they can split up into

(a) two teams of six

(b) three teams of four.

Problem 8

There are eight boys and five girls who attend the Senior Mathematics Club. Find how many ways the teacher can choose a team of six students to represent the school competition if:
(a) There are no gender restrictions.
(b) The team is to be made up of three girls and three boys.
(c) At least two of each gender are included in the team.

Problem 9

A four-digit number is made from the digits $0, 1, 3, 4, 5, 8, 9$.
(a) How many four-digit numbers can be made using the digits if repeats are allowed?
(b) How many of the four-digit numbers have no repeated digits?
(c) How many four-digit even numbers can be made using the digits?
(d) How many of these even numbers are divisible by 5?

Problem 10

A group of 12 people want to go to a concert. They can travel in a small car that takes one driver and one passenger and two cars each taking one driver and four passengers. If there are five drivers in the group, in how many different ways can they travel?

Solutions

Lesson 1.4HL Advanced Counting Principles

Warm Up

1(a) Divisible by both 2 and 3 means divisible by 6:

$$6, 12, 18 \quad \Rightarrow \quad \boxed{3}.$$

1(b) There are 10 multiples of 2 and 6 multiples of 3. Three are counted twice:

$$10 + 6 - 3 = \boxed{13}.$$

2) The letters can be arranged in $5!$ ways and the digits in $3!$ ways:

$$5! \times 3! = 120 \times 6 = \boxed{720}.$$

Applying Combination Solving Techniques

(a) Treat Ina and Angel as one block. There are six objects to arrange, and the pair can switch places:

$$6! \times 2! = \boxed{1440}.$$

(b) Subtract arrangements where they are together from all arrangements:

$$7! - 2(6!) = 5040 - 1440 = \boxed{3600}.$$

Separating Elements

First arrange the other four people:

$$4!.$$

This creates five gaps. Choose three gaps for Richard, Henry and Kevin and arrange them:

$$4!\binom{5}{3}3! = 24(10)(6) = \boxed{1440}.$$

Example 9

There are 10 students who are not from AIHL.
(a) Arrange those 10 students first:

$$10!.$$

They create 11 gaps. Choose 3 gaps for the three distinct AIHL students and arrange them:

$$10!\binom{11}{3}3! = \boxed{3\,592\,512\,000}.$$

(b) Treat the three AIHL students as one block. There are then 11 objects:

$$11! \times 3! = \boxed{239\,500\,800}.$$

Challenge Problems 1–4

Problem 1
The consonants are C, H, N, G, D. Choose and arrange two different consonants for the first and last positions:

$$5 \times 4.$$

Arrange the remaining five letters:

$$5 \times 4 \times 5! = \boxed{2400}.$$

Problem 2
Count all committees and subtract those containing both youngest students:

$$\binom{28}{5} - \binom{26}{3} = 98\,280 - 2\,600 = \boxed{95\,680}.$$

Problem 3
Subtract all-boy and all-girl teams from all teams:

$$\binom{19}{7} - \binom{9}{7} - \binom{10}{7}$$

$$= 50\,388 - 36 - 120 = \boxed{50\,232}.$$

Problem 4(a)

$$\binom{15}{6} = \boxed{5005}.$$

Problem 4(b)(i)

$$\binom{14}{6} = \boxed{3003}.$$

Problem 4(b)(ii)

$$\sum_{g=2}^{6}\binom{6}{g}\binom{9}{6-g} = \boxed{4165}.$$

Problem 4(c)

$$P(\text{at least 2 girls}) = \frac{4165}{5005} = \boxed{\frac{119}{143}}.$$

Challenge Problems 5–7

Problem 5(a)

If the largest card is 5, choose the other three from 1–4. If it is 6, choose from 1–5; if it is 7, choose from 1–6:

$$\binom{4}{3}+\binom{5}{3}+\binom{6}{3}=4+10+20=\boxed{34}.$$

Problem 5(b)

There are four even and five odd cards. Count selections with 2, 3 or 4 evens:

$$\binom{4}{2}\binom{5}{2}+\binom{4}{3}\binom{5}{1}+\binom{4}{4}\binom{5}{0}$$

$$=60+20+1=\boxed{81}.$$

Problem 6(a)

Choose six of the ten seats and arrange the six people:

$$\binom{10}{6}6!=\boxed{151\,200}.$$

Problem 6(b)

Choose a seat for the person with the cold, then choose five seats not adjacent to that seat and arrange the other five people. Summing over the possible cold-person seats gives

$$\boxed{33\,600}.$$

Problem 7(a)

Choose six for one team, but divide by $2!$ because the two teams are unlabeled:

$$\frac{\binom{12}{6}}{2}=\boxed{462}.$$

Problem 7(b)

Divide the 12 friends into three unlabeled groups of four:

$$\frac{12!}{(4!)^3\,3!}=\boxed{5775}.$$

Challenge Problems 8–10

Problem 8(a)

$$\binom{13}{6} = \boxed{1716}.$$

Problem 8(b)

$$\binom{5}{3}\binom{8}{3} = 10(56) = \boxed{560}.$$

Problem 8(c)

Possible gender splits are $2G4B$, $3G3B$, and $4G2B$:

$$\binom{5}{2}\binom{8}{4} + \binom{5}{3}\binom{8}{3} + \binom{5}{4}\binom{8}{2}$$

$$= 700 + 560 + 140 = \boxed{1400}.$$

Problem 9(a)

The first digit has 6 choices and each remaining digit has 7:

$$6(7^3) = \boxed{2058}.$$

Problem 9(b)

$$6 \times 6 \times 5 \times 4 = \boxed{720}.$$

Problem 9(c)

For an even number the last digit is $0, 4,$ or 8. With repeats allowed:

$$6 \times 7 \times 7 \times 3 = \boxed{882}.$$

Problem 9(d)

An even number divisible by 5 must end in 0:

$$6 \times 7 \times 7 = \boxed{294}.$$

Problem 10

Assuming the two larger cars are distinct, choose the driver of the small car, its passenger, and the two ordered drivers of the larger cars:

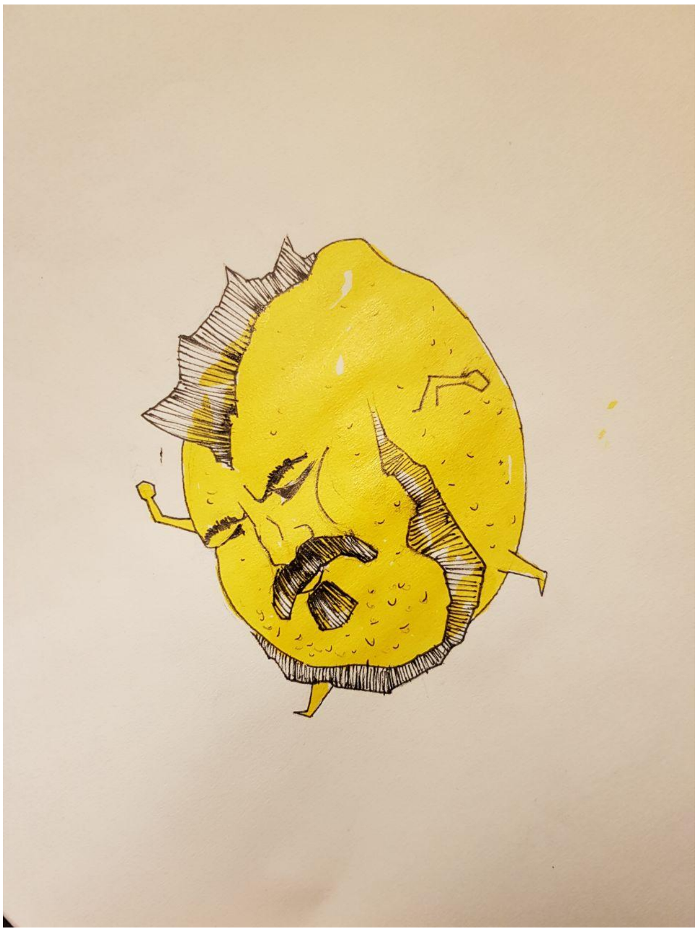
$$5 \times 11 \times 4 \times 3.$$

Then choose which four of the remaining eight passengers travel in the first large car:

$$\binom{8}{4}.$$

Therefore

$$5(11)(4)(3)\binom{8}{4} = \boxed{46\,200}.$$



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Mathematics made clear.