

DP Analysis & Approaches (HL)

Maths with Lemon

HL Topic 4 – Bayes' Theorem: Notes

AHL 4.13	Guidance, clarification and syllabus links
Bayes' theorem	Use of Bayes' theorem for a maximum of three events. Link to independent events (SL 4.6).

1. Conditional Probability Review

$$P(A | B) = \frac{P(A \cap B)}{P(B)}, \quad P(A \cap B) = P(A)P(B | A).$$

The event after the vertical bar is the information already known.

2. Derivation of Bayes' Theorem

Starting with

$$P(A | B) = \frac{P(A \cap B)}{P(B)}$$

and using $P(A \cap B) = P(A)P(B | A)$,

$$P(A | B) = \frac{P(A)P(B | A)}{P(B)}.$$

For the two exhaustive cases A, A' ,

$$P(B) = P(A)P(B | A) + P(A')P(B | A'),$$

so

$$P(A | B) = \frac{P(A)P(B | A)}{P(A)P(B | A) + P(A')P(B | A')}.$$

Bayes' theorem reverses the conditioning. In general, $P(A | B) \neq P(B | A)$.

3. Bayes' Theorem for Three Events

If A_1, A_2, A_3 are mutually exclusive and exhaustive,

$$P(B) = \sum_{j=1}^3 P(A_j)P(B | A_j),$$

and

$$P(A_i | B) = \frac{P(A_i)P(B | A_i)}{P(A_1)P(B | A_1) + P(A_2)P(B | A_2) + P(A_3)P(B | A_3)}.$$

4. Worked Example: Zoe

Suppose Zoe wins, draws or loses her first match with probabilities 0.6, 0.3, 0.1. Her probabilities of winning the tournament are 0.7, 0.4, 0.2, respectively. Let T mean she wins the tournament.

$$P(T) = 0.6(0.7) + 0.3(0.4) + 0.1(0.2) = 0.56.$$

Thus

$$P(\text{draw} | T) = \frac{0.3(0.4)}{0.56} = \frac{3}{14} \approx 0.214.$$

5. Tree-Diagram Thinking

Multiply probabilities along a path:

$$P(A_i \cap B) = P(A_i)P(B | A_i).$$

Add mutually exclusive paths to obtain the total probability of the evidence:

$$P(B) = \sum P(A_i \cap B).$$

Finally divide the required path by the total evidence.

6. Complements

Use

$$P(A') = 1 - P(A).$$

For example, if $P(D | H) = 0.15$, then

$$P(D' | H) = 0.85.$$

Words such as *not delayed*, *did not walk*, and *does not study biology* signal complement events.

7. Bayes with Frequencies

For a population of 500 cyclists containing 50 substance users:

$$50(0.45) = 22.5$$

expected true positives, while

$$450(0.085) = 38.25$$

expected false positives. Hence

$$P(\text{user} | +) = \frac{22.5}{22.5 + 38.25} = \boxed{0.370}.$$

This illustrates why the **base rate** matters.

8. Common Errors

- Reversing $P(A | B)$ and $P(B | A)$.
- Using only the numerator and forgetting total $P(B)$.
- Forgetting complements.
- Adding probabilities along a branch instead of multiplying.
- Omitting one possible cause of the observed event.
- Assuming independence without justification.

9. Problem-Solving Checklist

1. Define events.
2. Identify the observed evidence.
3. Calculate the probability of the required path.
4. Calculate the total probability of the evidence.
5. Divide using Bayes' theorem.
6. Interpret the result in context.



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Mathematics made clear.